

WHY LAGUERRE TESSELLATIONS ARE GOOD APPROXIMATIONS OF FOAMS

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Summary Stochastic models are valuable tools to study the mechanical behaviour of low-density foams. Owing to a similar cellular morphology, Laguerre tessellations have turned out to be the natural model for such foams. Laguerre tessellations are additively weighted generalisations of Voronoi tessellations, where polyhedral cells are formed through interaction of spheres. However, the cell curvature of foams allows only an approximation by Laguerre tessellations. This work studies the error in elasticity induced by the approximation based on simulated random foams. For low polydispersities, Young's modulus is overpredicted with an almost constant error of at most 5 % by the Laguerre approximations. With increasing polydispersity the prediction error increases following a cubic function.

INTRODUCTION

Low-density foams can be perceived as a subdivision of space into random (slightly curved) polyhedral cells whose edges form an interconnected network. Its local structure obeys Plateau's laws, that is, only three faces may intersect at a dihedral angle of exactly 120°. In stochastic geometry a subdivision of space with this topology is known as normal tessellation. It can be shown that any normal tessellation of the three-dimensional Euclidean space \mathbb{R}^3 is a Laguerre tessellation [1].

A Laguerre tessellation is an additively weighted generalisation of the well known Voronoi tessellation. The latter is generated by a locally finite set $\varphi = \{x_1, x_2, \dots\} \subset \mathbb{R}^3$ by assigning to each point $x \in \varphi$ the polyhedron $C(x, \varphi)$ composed of those points in space that have x as nearest neighbour in φ . To generalise this notion, we attach to the points of φ positive weights. Each pair $[x, r] \in \varphi$ can then be interpreted as a sphere with centre x and radius r , whose corresponding *Laguerre polyhedron* is defined by

$$C([x, r], \varphi) = \{y \in \mathbb{R}^3 : \|y - x\|^2 - r^2 \leq \|y - x'\|^2 - r'^2 \text{ with } [x', r'] \in \varphi\}. \quad (1)$$

If all radii are equal, we obtain the Voronoi polyhedron as special case.

Random Laguerre tessellations appear to be the natural model for low-density foams as both share the same topology and a similar random cellular morphology. However, the cell curvature of foams allows only an approximate representation by Laguerre tessellations. To answer the question how good such an approximation can be, we follow the approach in [2]. Its main idea is to find a set of spheres φ whose Laguerre tessellation best fits the cell system (and thus the edge system) of the foam by minimising the discrepancy between the individual cells of the foam and the Laguerre tessellation. We call such a tessellation *Laguerre approximation*.

The results in [2] show that the cellular morphology as well as the local topology are reproduced quite well by Laguerre approximations. In this work, we focus on the error in elasticity induced by the approximation based on simulated random foams. The foams were produced with the Surface Evolver [3] by minimising the surface area of Laguerre polyhedra generated from dense polydisperse sphere packings. We then computed the Laguerre approximations of the foams, calculated Young's moduli of the resulting tessellations by finite element analysis, and compared them to the moduli of the foams. The result of this comparison is shown in Figure 1.

ELASTICITY OF APPROXIMATED OPEN CELL FOAMS

For our study we used 8 foams with 1728 and 51 foams with 2197 cells. Their polydispersities ranged from 0 to 0.6, whereas values up to 0.3 are commonly found in real foams. The polydispersity p is defined as the quotient of the standard deviation σ_r and the mean \bar{r} of the equivalent sphere radius r , i. e. $p = \sigma_r / \bar{r}$, where r is determined from the volume $V = \frac{4}{3}\pi r^3$ for each cell of the foam. Note that p equals zero only for monodisperse foams.

The foams were approximated using the method presented in [2]. This method provides an exact reconstruction of a tessellation if it is known to be a Laguerre tessellation. However, as foams cells (especially in foams with high polydispersity) have curved surfaces, an exact representation by Laguerre polyhedra is impossible. To measure the goodness-of-fit of the approximation, we compare the individual foam cells with their approximations and count the number of extra or missing cell neighbours. We denote this quantity as $\mathcal{F}_{\text{diff}}$.

The lower part of Figure 1 shows the mean goodness-of-fit of the Laguerre approximations, grouped into eight classes with respect to polydispersity. Between a polydispersity of 0 and 0.45 the approximation error stays below 5 % with a minimum of about 1 % around $p \approx 0.2$. For polydispersities above 0.45 the error increases to about 14 % at $p \approx 0.6$. This result is not surprising as Plateau's laws have great influence on the morphology of foams, whose cells necessarily need curvature to maintain their local topology.

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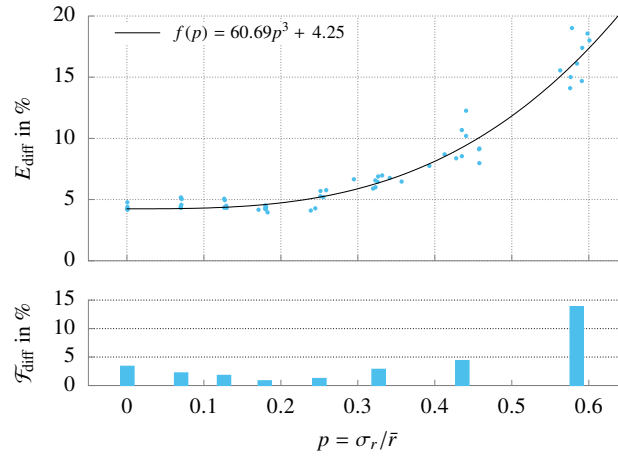


Figure 1: Increase in Young's modulus E_{diff} when approximating foams by Laguerre tessellations with respect to polydispersity p . T_{diff} is the mean error in local topology between a foam and its approximation, that is, the number of extra or missing cell neighbours.

To study the influence of the approximation error on elasticity, we determined the difference E_{diff} of Young's modulus for each foam and its corresponding Laguerre approximation. In both cases, Young's modulus was computed by connecting the vertices with straight beam elements of uniform Plateau-border cross section. The Young's moduli of the foams were always softer than then the ones of the Laguerre approximations. This effect is caused by the definition of faces in Laguerre polyhedra, where vertices must lay in a common plane. Owing to cell curvature this restriction does not hold for foams.

E_{diff} is depicted in the upper part of Figure 1. Up to a polydispersity of about 0.25, the error induced in Young's moduli by the Laguerre approximations is relatively insensitive with respect to polydispersity. Within this range it is almost constant and smaller than 5 %. The error then rapidly increases to about 17 % with increasing polydispersity. This relation is well described by a cubic function of the polydispersity p , more precisely

$$f(p) = 60.69p^3 + 4.25. \quad (2)$$

The coefficients of Equation (2) were determined by regression analysis using weighted least squares.

CONCLUSIONS

It was shown in [2] that the cellular morphology of foams is reproduced quite well by Laguerre approximations. For polydispersities below 0.45, the lower part of Figure 1 illustrates that even the local topology of the approximations is in astonishing agreement with the simulated foams. Within this range we observed at most 5 % incorrectly assigned cell neighbours in the complete structure. Hence, we may conclude that from a geometric point of view Laguerre tessellations are good approximations for low- to mid-polydisperse low-density foams.

In the presented study we additionally considered the mechanical response of open cell foams. For low-polydisperse foams with $0 \leq p < 0.25$, we found the elastic behaviour of the Laguerre approximations in very good agreement with the one of the simulated foams. The Young's moduli of the approximations were at most 5 % stiffer than the ones of the simulated foams. For mid-polydisperse foams with $0.25 \leq p < 0.5$, the error in Young's modulus increases to 12 % following a cubic function, despite the good geometric approximation. This result is not surprising as with increasing polydispersity the influence of cell curvature on the elastic behaviour becomes predominant.

In practice, however, foams barely exceed a polydispersity of 0.3. Laguerre tessellations are then able provide realistic models for low-density (open cell) foams. Even for higher densities Laguerre tessellations provide reasonable models for the mechanical behaviour of foams [4]. Moreover, the approach presented in this work can be applied to establish bounds for certain mechanical properties of foams as the isotropic Plateau polyhedra theory does for geometric ones [5]. An according study is subject of ongoing research.

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