An Alternative to the "2-Column" Proof

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Today's Talk

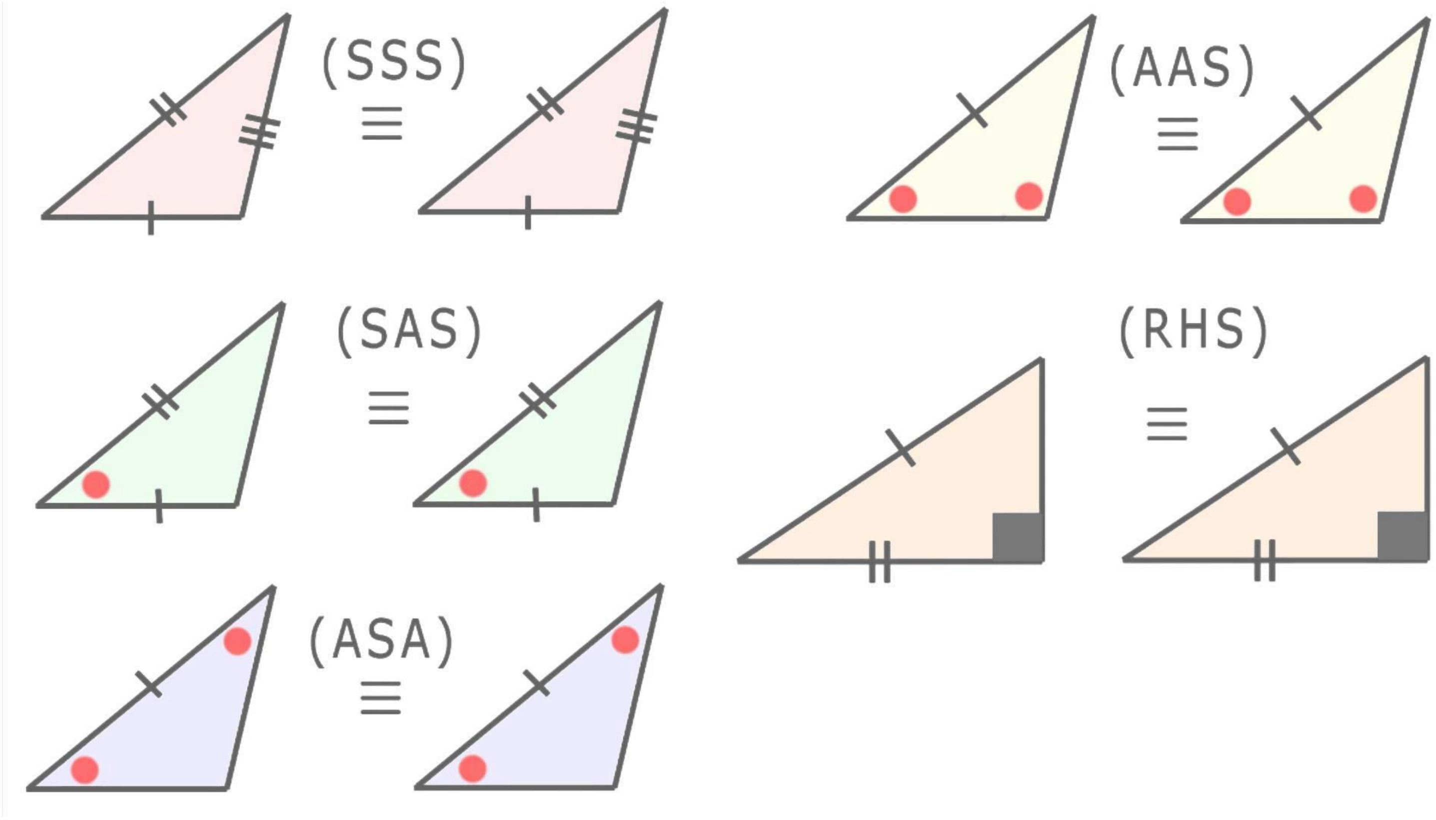
- The classic "2-column proof"
- What do we want and expect from proofs?
- Demonstrate a different way of proving statements

- Do you know what math holiday is today?
  - Pi Day (March 14 = 3-14 or 3.14)
- And also… in 1879, this famous German-American physicist was born…
  - Albert Einstein!

Before we get started…

- Who is here today?
- What school (or what district)?
- What do you teach?
- What grades?

Do we know this stuff? If not, it's okay.



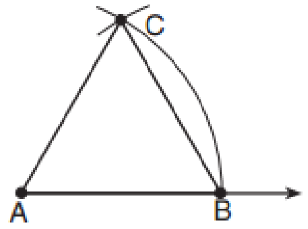
A little about me and my school…

- Eric has taught high school math at the Gow School for 12 years, mainly teaching Geometry, Algebra 2, Trigonometry, and Computer Science.
- The Gow School is a college preparatory boarding and day school for students, grades 6-12, with dyslexia and similar language-based learning disabilities.
- Learn more @ www.gow.org
- Gow is accredited by the New York State Association of Independent Schools.
- That’s awesome, Eric. But… who cares?

Well the thing is…

- Common Core and NYS Regents tests do not occur at the Gow School.

9 The diagram below shows the construction of an equilateral triangle.



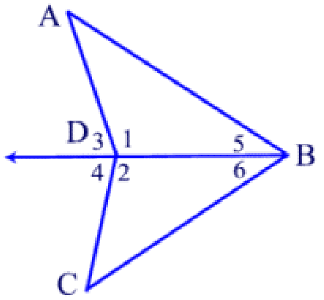
Which statement justifies this construction?

(1) $\angle A + \angle B + \angle C = 180$ (3) $AB = AC = BC$

(2) $m\angle A = m\angle B = m\angle C$ (4) $AB + BC > AC$

From RegentsPrep.com

Question #5



Given: $\overline{BD}$

$\overline{AD} \cong \overline{CD}$

$\measuredangle 3 \cong \measuredangle 4$

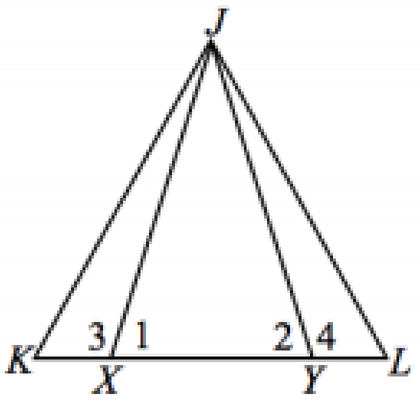
Prove: $\overline{DB}$ bisects $\measuredangle ABC$

	Statements	Reasons
1.	$\overline{BD}$ $\overline{AD} \cong \overline{CD}$ $\measuredangle 3 \cong \measuredangle 4$	1. Given
2.	$\measuredangle 1$ supp $\measuredangle 3$ $\measuredangle 2$ supp $\measuredangle 4$	2. Two angles that form a linear pair are supplementary.
3.	$\angle 1 \cong \angle 2$	3. Supplements of the same angle, or congruent angles, are congruent.
4.	$\overline{DB} \cong \overline{DB}$	4. Reflexive property: A quantity is congruent to itself.
5.	$\Delta ABD \cong \Delta CBD$	5. (SAS) If two sides and the included angle of one triangle are congruent to the corresponding parts of a second triangle, the triangles are congruent.
6.	$\measuredangle 5 \cong \measuredangle 6$	6. (CPCTC) Corresponding parts of congruent triangles are congruent.
7.	$\overline{DB}$ bisects $\measuredangle ABC$	7. An angle bisector is a ray whose endpoint is the vertex of the angle and which divides the angle into two congruent angles.

EngageNY.org

3. **Given:** $JX = JY$, $KX = LY$

Prove: $\triangle JKL$ is isosceles



$JX = JY$	*Given*
$KX = LY$	*Given*
$m\angle 1 = m\angle 2$	*Base angles of an isosceles triangle are equal in measure.*
$m\angle 1 + m\angle 3 = 180°$	*Linear pairs form supplementary angles.*
$m\angle 2 + m\angle 4 = 180°$	*Linear pairs form supplementary angles.*
$m\angle 3 = 180° - m\angle 1$	*Subtraction property of equality*
$m\angle 4 = 180° - m\angle 3$	*Subtraction property of equality*
$m\angle 3 = m\angle 4$	*Substitution property of equality*
$\therefore \triangle JKX \cong \triangle JYL$	*SAS*
$\overline{JK} \cong \overline{JL}$	*Corresponding parts of congruent triangles are congruent.*
$\therefore \triangle JKL$ *is isosceles.*	*Definition of an isosceles triangle*

From MathWithBadDrawings.com

Theorem: Two-column proofs are **great* preparation for the future.

Statement	Reason
1. High school ought to prepare students for their remaining years as scholars, and their future decades as citizens.	1. Definition of Education
2. Real mathematicians employ two-column proofs all the time!	2. Theorem of Lies
3. Two-column proofs are also used in the workplace and the political sphere. They're everywhere!	3. Theorem of Even More Lies
4. I mean, we wouldn't be using such an artificial, opaque system for teaching logic if it didn't have *some* real-world utility, right?	4. Definition of Wishful Thinking
5. Logic ought to be learned through two-column proofs.	5. Axiom of Systemic Stubbornness in Education

Another *Proof* from MathWithBadDrawings.com

Theorem: "Justifying steps" out to be an opaque, frustrating process.

Statement	Reason
1. In an argument, all steps must be justified.	1. Definition of Argument
2. In real, adult arguments, such justifications often take the form of cogent explanations, appeals to agreed-upon facts, and clear, explicit reasoning.	2. Definition of Justification
3. High schoolers are too simpleminded for such techniques.	3. Fundamental Axiom of Condescension Towards Young People
4. Besides, it would take too long for instructors to grade such arguments.	4. Overworked Teacher Postulate
5. Instead, high schoolers ought to justify their arguments by reciting the names of theorems and axioms, invoked as if they were not logical statements but magical spells.	5. Property of Nonsensical Schooling
6. Logic ought to be learned through two-column proofs.	6. Theorem of Maximal Damage

What should we expect from our students' proofs anyway?

- Here's what NCTM says about *Reasoning & Proof*:

Instructional programs from prekindergarten through grade 12 should enable all students to –

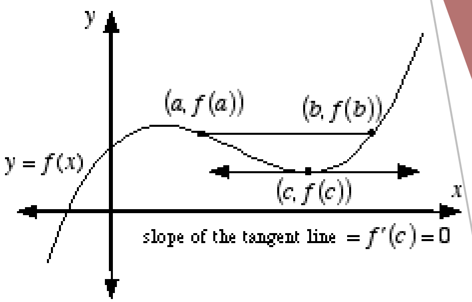
  - Recognize reasoning and proof as fundamental aspects of mathematics
  - Make and investigate mathematical conjectures
  - Develop and evaluate mathematical arguments and proofs
  - Select and use various types of reasoning and proof

NCTM.org, *Principles and Standards for School Mathematics: Standards and Positions* (accessed Feb 2017).

NCTM also says…

- Instructional programs from prekindergarten through grade 12 should enable all students to –
  - Organize and consolidate their mathematical thinking through communication (*Communication*)
  - Communicate their mathematical thinking coherently and clearly to peers, teachers, and others (*Communication*)
  - Use the language of mathematics to express ideas precisely (*Communication*)
  - Understand how mathematical ideas interconnect and build on one another to produce a coherent whole (*Connections*)
  - Create and use representations to organize, record, and communicate mathematical ideas (*Representation*)

Standard Proof from Calculus AB



- Rolle's Theorem: Let $f(x)$ be a function which is continuous on the closed interval $[a,b]$ and differentiable on every point of the interior $[a, b]$. Suppose that $f(a) = f(b)$. Then there is a c between a and b where $f'(c) = 0$.

By the extreme value theorem, f achieves its maximum on $[a,b]$. By applying the extreme value theorem to $-f$, we see that f also achieves it minimum on $[a,b]$. By hypothesis, if both the maximum and minimum are achieved on the boundary, then the maximum and minimum are the same and thus the function is constant. A constant function has zero derivative everywhere. If f is not constant, then f has either a local minimum or a local maximum in the interior. The derivative at a local minimum or maximum must be zero.

A little more algebraic…

- Suppose there are finitely many primes: p_1, p_2, p_3, … , p_n. Consider the value:

$$p = p_1 \cdot p_2 \cdot p_3 \cdot \dots \cdot p_n + 1$$

- Clearly, p is larger than any of the primes, so it doesn't equal any of them. And since p_1, p_2, p_3, … , p_n constitute all of the primes, p can't be prime.
- Thus, p must be divisible by one of the primes in the list – say p_m.
- So:

$$\frac{p}{p_m} = \frac{p_1 \cdot p_2 \cdot p_3 \cdot \dots \cdot p_n}{p_m} + \frac{1}{p_m}$$

- Since p_m doesn't divide into 1 evenly, the original assumption must be false.

Maybe something from Algebra 1?

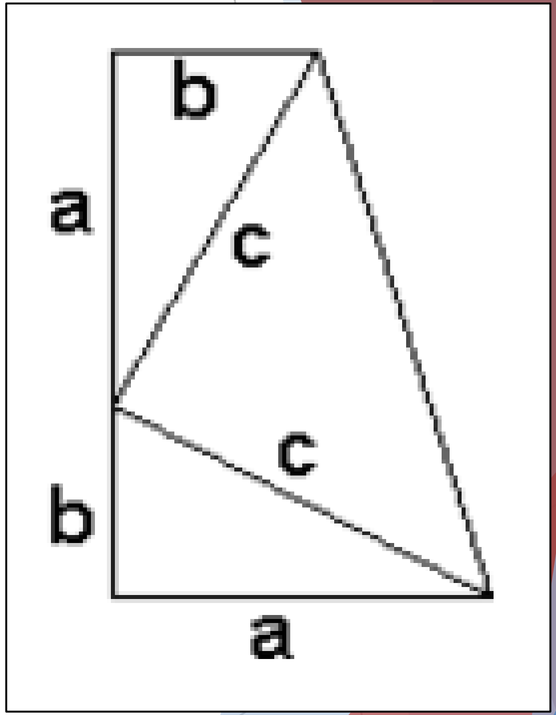
The diagram at right represents two copies of a right triangle, with legs *a* and *b* and hypotenuse *c*. Notice that the area can be expressed as a trapezoid or as three triangles.

$$\frac{a+b}{2}\cdot(a+b)=\frac{1}{2}ab+\frac{1}{2}ab+\frac{1}{2}c^2$$

$$(a+b)(a+b)=ab+ab+c^2$$

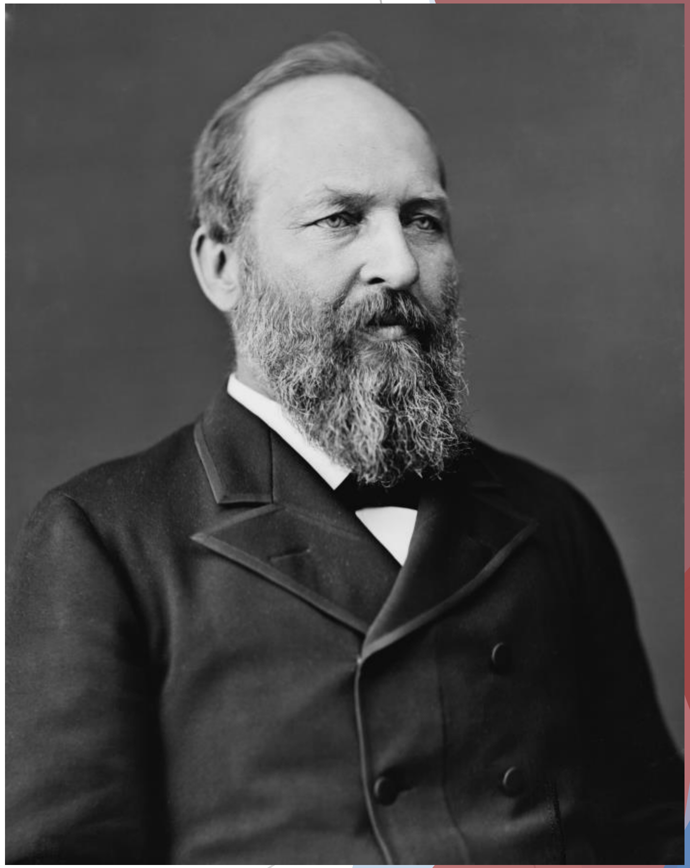
$$a^2+ab+ab+b^2=ab+ab+c^2$$

$$a^2+b^2=c^2$$



Who came up with the proof on the previous slide?

- James Garfield in 1876
- Garfield was the 20th president of the USA (March-September, 1881)



Where did 2-column proofs come from anyways?

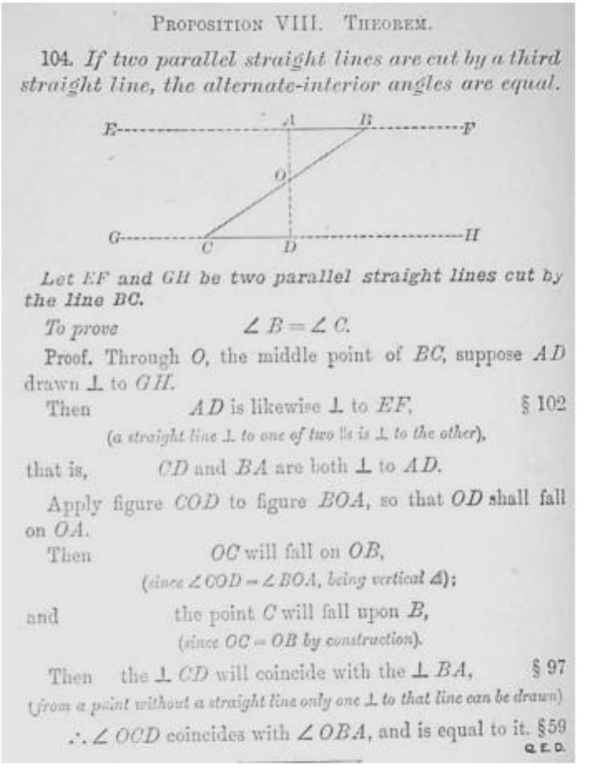
- In the 1880s, there were no standards in place for secondary schools and there was a general dissatisfaction of secondary education in the US.
- In response, the National Education Association appointed the “Committee of Ten” to standardize school curricula
- The Committee of Ten appointed a sub-committee on mathematics
- The subcommittee held a 3-day meeting at Harvard University in December 1892 and filed a report in March 1893. Recommendations included:
  - Teaching should “exercise the pupil’s mental activity” and “rules should be derived inductively instead of dogmatically.”
  - And “geometrical demonstration is to be chiefly prized [as] a discipline in complete, exact, and logical statement.”

And then…

- Prior to this, proofs written in geometry textbooks were written in paragraph format and not many details were provided.
- In 1878, George Wentworth's *Elements of Geometry* began to structure what would become the 2-column proof.
- In 1899, Wooster Beman and David Smith started a series of *Plane Geometry* that thoroughly described the nature of logical proof, indicating that:

  "Every statement in a proof must be based on a postulate, an axiom, a definition, or some proposition previously considered."

- In 1913, the first 2-column proof appeared in Arthur Schultze and Frank Sevenoak's second edition geometry book.



Proposition VIII. Theorem.

104. *If two parallel straight lines are cut by a third straight line, the alternate-interior angles are equal.*

Let EF and GH be two parallel straight lines cut by the line BC.

To prove $\angle B = \angle C$.

Proof. Through O, the middle point of BC, suppose AD drawn ⊥ to GH.

Then AD is likewise ⊥ to EF, § 102

(*a straight line ⊥ to one of two ∥s is ⊥ to the other*).

that is, CD and BA are both ⊥ to AD.

Apply figure COD to figure BOA, so that OD shall fall on OA.

Then OC will fall on OB,

(*since* $\angle COD = \angle BOA$, *being vertical* △);

and the point C will fall upon B,

(*since* $OC = OB$ *by construction*).

Then the ⊥ CD will coincide with the ⊥ BA, § 97

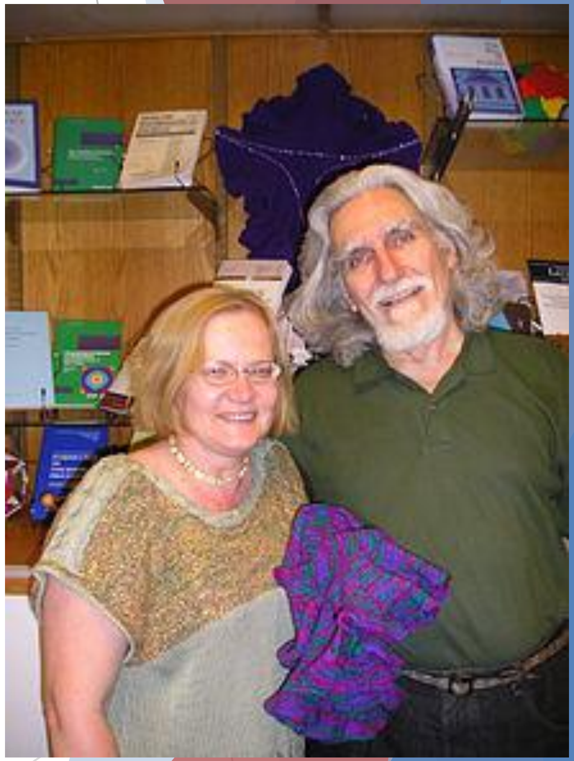
(*from a point without a straight line only one ⊥ to that line can be drawn*)

∴ $\angle OCD$ coincides with $\angle OBA$, and is equal to it. § 59

Q.E.D.

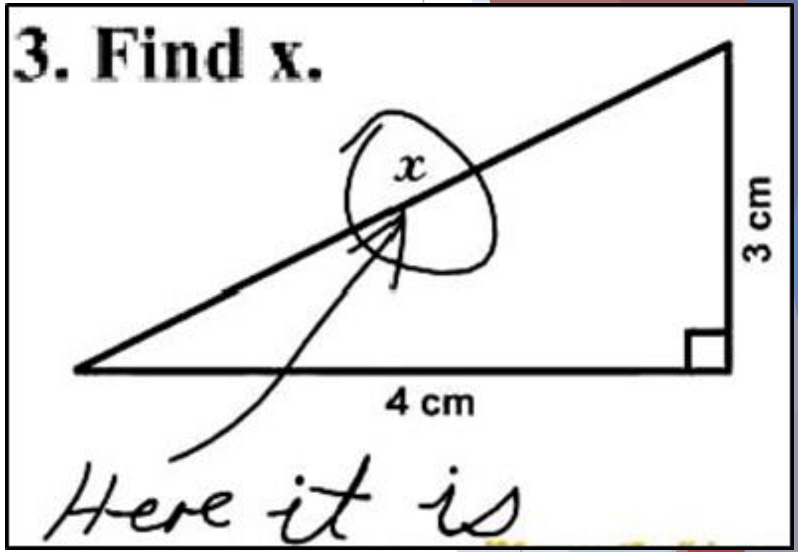
Dr. David Henderson, Ph.D.
Professor Emeritus of Mathematics, Cornell University

- Authored "Proof as a Convincing Communication That Answers – Why?"
  - Argued that proofs needed to be humanized, which means they should:
    - Communicate clearly to the appropriate audience.
    - Convince the reader.
    - Completely answers questions like: "Why?" or "Where did it come from?"
- David formerly instructed *Math 451: Euclidean, Spherical, and Hyperbolic Geometry* (among other classes) at Cornell. Topics of Math 451 included:
  - When do you call a line straight?
  - What is an angle?
  - Euclid's 5th postulate.



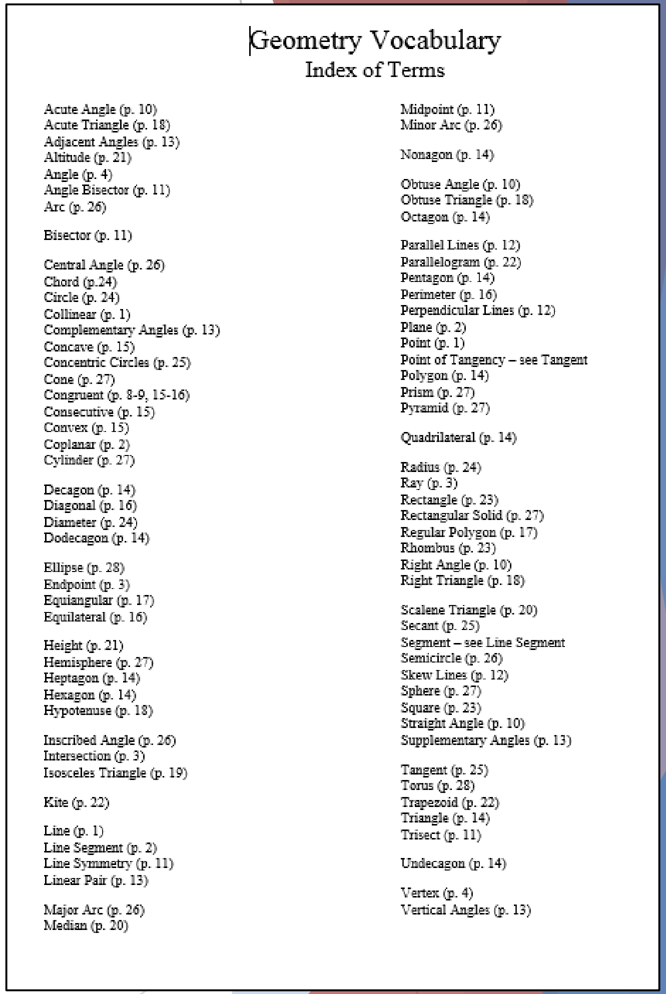
Eric's Sequence of Topics in Geometry

- MP 1 (Sept-Oct): Inductive Reasoning
  Geometry Vocabulary
- MP 2 (Oct-Dec): Compass & Straighedge Constructions
  Line/Angle Properties
  Triangle Properties
- MP 3 (Jan-Mar): Deductive Reasoning (Proofs)
  Polygon Properties
  Circle Properties
- MP 4 (Mar-May): Everything else!



Teaching the Vocabulary

- I actually have an entire 60-minute presentation on this topic.
- Real quick though…
  - Plan as much as you can!
    - Lots of examples, diagrams, mnemonics, etc.
    - Make sure you give a thorough look through EVERY term you need
  - Do not rush instruction.
  - Quiz regularly
    - 2-3 short quizzes per week during instruction (<5 minutes)
  - Reinforce as much as you can as often as you can.
    - Projects, papers, and oral quizzing



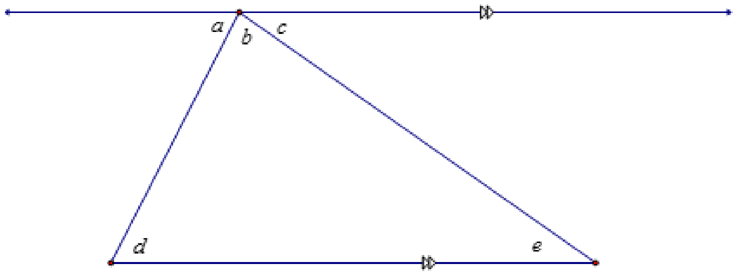
Geometry Vocabulary

Index of Terms

Giving it an optional try…

BONUS PROBLEM

Write a paragraph proving that the interior angle sum of a triangle is 180°.



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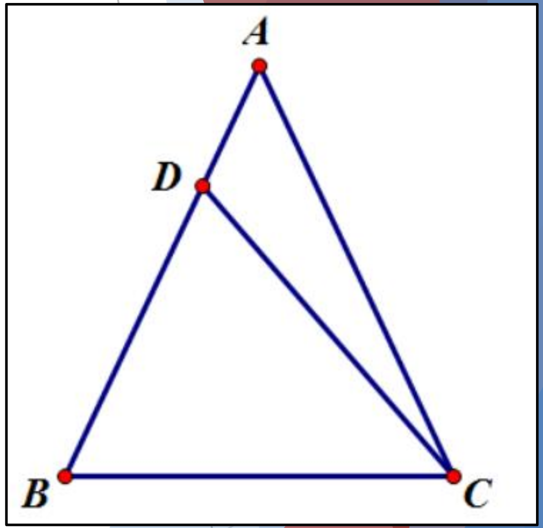
Talking about Euclid's *The Elements*

- Book 1, Proposition 6: *If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.*

Let ABC be a triangle with $\angle ABC \cong \angle ACB$. I say that AB = AC. If AB does not equal AC, then one of them is greater – say AB is greater.

Cut off DB from AB (the greater) so DB = AC and join DC. Since DB = AC, then DB/BC = AC/CB and the angle DBC = angle ACB. So, $\Delta DBC \cong \Delta ACB$ because SAS.

Therefore DC = AB. This means the lesser length equals the greater. Absurd! So AB cannot be unequal to AC. It must be that AB = AC.



Basic Sentence Patterns (BSPs)

BSP #1

If <*given fact*>, then <*logical conclusion*> because <*explanation*>.

Example 1: M is the midpoint of line segment AB

If M is the midpoint of line segment AB, then AM = BM because a midpoint divides a line segment into two congruent parts.

Example 2: m<TOP = 45° and m<GUN = 45°

Some more BSPs

BSP #2 – for GIVEN statements

It is given that <*given fact*>.

BSP #3 – for the REFLEXIVE PROPERTY

Triangle <*one*> and triangle <*two*> share <common part>.

BSP #4 – stating that two TRIANGLES are CONGRUENT

Therefore, <*two triangles are congruent*> because <*SSS, SAS, ASA, AAS, or HL*>.

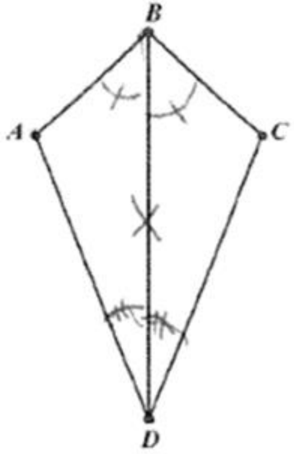
BSP #5 – Corresponding Parts of Congruent Triangles are Congruent

<*Part 1*> is congruent to <*Part 2*> because CPCTC.

An example of a "good one"

2. Given: $\overline{BD}$ bisects $\angle ABC$ ✓
$\overline{DB}$ bisects $\angle ADC$

Prove: $\overline{AB} \cong \overline{CB}$



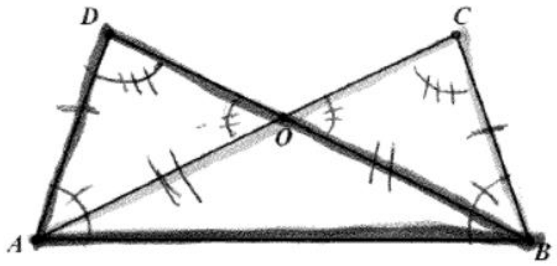
ASA

If $\overline{BD}$ bisects $\angle ABC$, then $\angle ABD \cong \angle CBD$ because a bisector divides an angle in half. If $\overline{DB}$ bisects $\angle ADC$, then $\angle ADB \cong \angle CDB$ because a bisector divides an angle in half. $\triangle ABD$ and $\triangle CBD$ share $\overline{BD}$. Therefore $\triangle ABD \cong \triangle CBD$ because ASA. $\overline{AB} \cong \overline{CB}$ because CPCTC.

Here's a longer one…

- Given: $\overline{DA} \cong \overline{CB}$
  $\angle DAB \cong \angle CBA$

  Prove: ΔAOB is isosceles



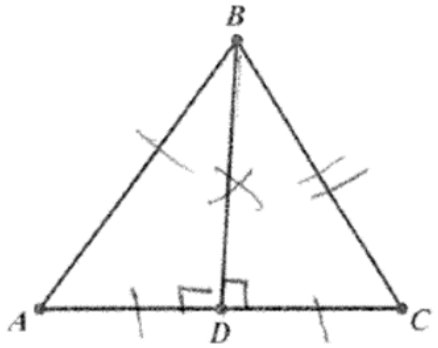
It is given that $\overline{DA} \cong \overline{CB}$ and $\angle DAB \cong \angle CBA$. ΔDAB and ΔCBA share $\overline{AB}$. Therefore $\Delta DAB \cong \Delta CBA$ because SAS.

$\angle D \cong \angle C$ because CPCTC. If $\overline{DB}$ and $\overline{CA}$ intersect at O, then $\angle COB \cong \angle DOA$ because they are vertical angles. Therefore $\Delta COB \cong \Delta DOA$ because AAS. $\overline{AO} \cong \overline{BO}$ because CPCTC. If $\overline{AO} \cong \overline{BO}$, then ΔAOB is isosceles because an isosceles triangle has two equal sides.

Proof by Contradiction

3. Given: $AB \neq BC$
$\overline{BD}$ is a median ←

Prove: $\overline{BD}$ is not perpendicular to $\overline{AC}$

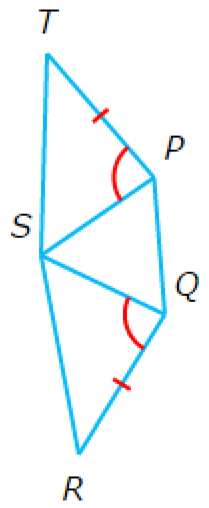


SAS

Suppose $\overline{BD} \perp \overline{AC}$. If $\overline{BD} \perp \overline{AC}$, then $\angle BDA \cong \angle BDC$ because perpendicular lines make right angles. If $\overline{BD}$ is a median, then $\overline{AD} \cong \overline{CD}$ because a median divides the base in half. $\triangle BAD$ and $\triangle BCD$ share $\overline{BD}$. Therefore $\triangle BAD \cong \triangle BCD$ because SAS. $\overline{AB} \cong \overline{BC}$ because CPCTC. However, it is given that $AB \neq BC$. Therefore $\overline{BD}$ is not perpendicular to $\overline{AC}$.

Using a 2-Column Proof to Help Write Your Paragraph

ΔPQS is equilateral. Complete the proof that $\Delta QRS \cong \Delta PTS$.



	Statement	Reason
1	ΔPQS is equilateral	Given
2	$\overline{QR} \cong \overline{PT}$	Given
3	$\angle SPT \cong \angle RQS$	Given
4	$\overline{PS} \cong \overline{QS}$	Definition of equilateral triangle
5	$\Delta QRS \cong \Delta PTS$	SAS

If ΔPQS is equilateral, then $\overline{PS} \cong \overline{QS}$ because all sides are congruent in an equilateral triangle. It is given that $\overline{QR} \cong \overline{PT}$ and $\angle SPT \cong \angle RQS$. Therefore, $\Delta QRS \cong \Delta PTS$ because SAS.

Once they're comfortable, let them go on their own.

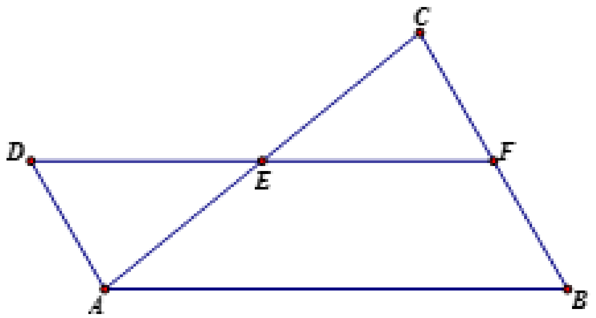
- Inevitably, the question of "Do I have to use the BSPs?" comes up.
  - My response: Are you proving the statement?
- Collaboration can be good. Have two students work on one.
  - Cuts your work load in half!
  - Usually style/neatness is better.
- Peer reviewing is helpful. Have students ask:
  - Is it written clearly? Concisely?
  - Has it been accurately proven? Are they convinced?
  - Can it be explained verbally?

Try one on your own…

2. Complete the proof below by writing a paragraph.

Given: $\overline{AC}$ and $\overline{DF}$ bisect each other at E
$\overline{AD} \cong \overline{CF}$

Prove: $\Delta DEA \cong \Delta FEC$



How'd you do?

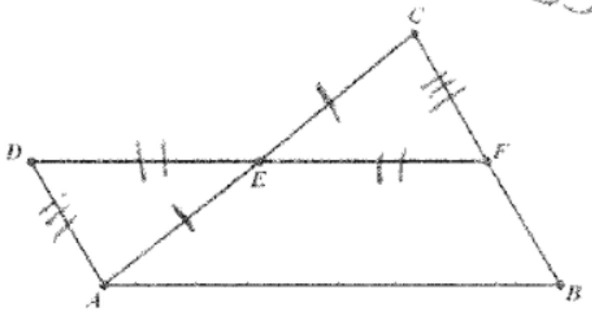
SSS

2. Complete the proof below by writing a paragraph.

Given: $\overline{AC}$ and $\overline{DF}$ bisect each other at E

$\overline{AD} \cong \overline{CF}$

Prove: $\Delta DEA \cong \Delta FEC$



If $\overline{AC}$ and $\overline{DF}$ bisect each other at E, then $\overline{AE} \cong \overline{CE}$ and $\overline{DE} \cong \overline{EF}$ because a bisector divides a segment in 2 equal parts. It is given that $\overline{AD} \cong \overline{CF}$. Therefore, $\Delta DEA \cong \Delta FEC$ because SSS.

Questions?

Thanks!

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